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Feynman path integral for the Dirac equation

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Abstract. The evolution operator for the Dirac equation is expressed as a sequential Feynman path integral. The usual approximations in terms of integrals over finite-dimensional spaces have been explicitly calculated in this paper and written in terms of translation operators (see equation (20)). This expression should lend itself well to numerical approximations.

1. Introduction

In this paper we shall express, by means of a type of Feynman path integral, the solution of the Dirac differential equation

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(t,x)=\left[\phi(x)+\sum_{k=1}^{3}\alpha^k\left(-\mathrm{i}\frac{\partial}{\partial x_k}-a_k(x)\right)+\beta b\right]\Psi(t,x) \tag{1}$$

with $\Psi(0,x)=f(x)$ and where Ψ is defined on $\mathbb{R}^4$ (time–space (t, x_1, x_2, x_3)) with values in the spin space $(\Psi_1, \Psi_2, \Psi_3, \Psi_4)\in\mathbb{C}^4$. The vector (a_1, a_2, a_3) is proportional to the magnetic potential, ϕ to the electric potential, and b to the rest mass. The α^k $(k=1,2,3)$, $\beta=\alpha^4$ are 4×4 Hermitian matrices which satisfy the relation

$$\alpha^k\alpha^j+\alpha^j\alpha^k=2\delta^{kj}I \qquad \text{for } k,j=1,2,3,4. \tag{2}$$

For a representation of the matrices α^k, β see e.g. Messiah (1965).

We can construct the Feynman path integral in a similar form to the Wiener approximation scheme. We give an approximation to the integral over all the paths, $\gamma(t)$, by the integral over the polygonal paths with corners $(t_j, \gamma(t_j))$ where $0=t_0<t_1<\ldots<t_l=s$ is a partition of the time interval $(0,s)$, and we pass to the limit $\max(t_{j+1}-t_j)\to0$. In this approximation we deal with integrals over finite-dimensional spaces, instead of integrals over path spaces.

We will define the approximations $E(\pi,\tau)$ on $S(\mathbb{R}^n)$, the Schwartz space of C^∞ functions of rapid decrease, by

$$E(\pi,\tau)f(y)=G(\tau-t_k,y)G(t_k-t_{k-1},x^k)\ldots G(t_1-t_0,x^1)f \tag{3}$$

where $\pi=\{t_0,t_1,\ldots,t_l\}$ is a partition of $[0,s]$, $\tau\in(t_k,t_{k+1}]$, $f\in S(\mathbb{R}^n)$ and

$$G(t,y)f=(2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\phi(y)t}\,\mathrm{e}^{-\mathrm{i}\beta bt}\prod_{k=1}^{3}\exp[\mathrm{i}\alpha^k a_k(y)t]$$
$$\times\prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t)\,\mathrm{e}^{\mathrm{i}p(y-x)}f(x)\,\mathrm{d}x\,\mathrm{d}p. \tag{4}$$

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Using a limiting procedure, the Plancherel theorem, we extend these definitions to $\mathscr{L}^2(\mathbb{R}^3)$. We shall go on to prove that $E(\pi, \tau)$ converges in $\mathscr{L}^2$ to the evolution operator for (1), when $\varepsilon(\pi) = \max(t_{j+1} - t_j)$ converges to zero, if ϕ, a_j are bounded functions of class C^1.

The convergence of these integrals can be proved in various ways, for instance, using the Kato–Trotter formula or an approach similar to Pliś (1976). The advantage of the proof in this paper is that a numerical approximation of the solution can be developed on the basis of equation (20).

The original idea to develop an approximation to the evolution operator for the Dirac equation, by integrating only along the paths with the velocity of light, was conceived by R P Feynman himself in the one-dimensional case. Feynman's approximation is different from ours, but our approximation is defined for the n-dimensional case and conserves the $\mathscr{L}^2$ norm (see the remark after lemma 3).

In Albeverio and Hoegh-Krohn (1976) and Proc. Colloq. on Feynman Path Integrals (1979) there is an up to date bibliography about Feynman path integral.

2. Notations and propositions

If $z \in \mathbb{R}^m$, $A \subset \mathbb{R}^m$ then $|z| = \max|z_i|$ and

$$B(z, \gamma) = \{w \in \mathbb{R}^m; |z - w| < \gamma\}, B(A, \gamma) = \bigcup_{z \in A} B(z, \gamma).$$

If $n = (n_1, n_2, n_3)$, where n_j are non-negative integer numbers, then $|n| = n_1 + n_2 + n_3$, and

$$D^n f = \partial^{|n|} f / \partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3} \qquad \text{for } x \in \mathbb{R}^3.$$

Let Ω be an arbitrary open set in $\mathbb{R}^3$. We shall employ the usual notations for the function spaces, with domain Ω and $\mathbb{C}^4$-valued: $C^m(\Omega)$, $C_0^m(\Omega)$ (functions of class C^m and compact support) and $\mathscr{L}^2(\Omega)$ with norm $\| \ \|$. Denote by $H^m(\Omega)$ the Sobolev space of order m on Ω

$$H^m(\Omega) = \{f : \Omega \to \mathbb{C}^4 \,|\, D^n f \in \mathscr{L}^2(\Omega), \text{for all } n, |n| \leq m\} \tag{5}$$

with the norm

$$\|f\|_m = \left(\sum_{|n| \leq m} \int_{\mathbb{R}^3} |D^n f|^2 \right)^{1/2}. \tag{6}$$

For instance $H^0(\Omega) = \mathscr{L}^2(\Omega)$ and we will denote $\| \ \|_0$ by $\| \ \|$.

The Fourier–Plancherel transformation $\mathscr{F}$ is defined by

$$\mathscr{F}(f(x)) = \hat{f}(y) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ixy} f(x)\, dx \qquad \text{for each } y \in \mathbb{R}^3. \tag{7}$$

We shall need the following known results (Maurin 1972).

Proposition 1. The Fourier–Plancherel transformation is an isometry in $\mathscr{L}^2$ with inverse

$$\mathscr{F}^{-1}(\hat{f}(y)) = f(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{ixy} \hat{f}(y)\, dy \qquad \text{for each } x \in \mathbb{R}^3. \tag{8}$$

Proposition 2. If $\bar{\Omega}$ is a compact set, the natural embedding $H^l(\Omega) \to H^k(\Omega)$, is a compact map, for $l > k$.

By virtue of (2), this is simple to prove.

Proposition 3. The Dirac matrices α^k, β satisfy

$$\exp(\mathrm{i}\alpha^k a) = I\cos a + \mathrm{i}\alpha^k \sin a \qquad \text{for } k=1,2,3,4;\ a\in\mathbb{R}, \tag{9}$$

$$\exp(\mathrm{i}\alpha^k a)\alpha^j = \alpha^j \exp(-\mathrm{i}\alpha^k a) \qquad \text{for } j,k=1,2,3,4;\ j\neq k;\ a\in\mathbb{R}. \tag{10}$$

In order to simplify the notation we need to define

$$\begin{aligned}
G^1(t,y)f &= (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\phi(y)t}\,\mathrm{e}^{\mathrm{i}\beta t}\prod_{k=1}^{3}\exp[\mathrm{i}\alpha^k a_k(y)t] \\
&\quad\times \prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t)\,\mathrm{e}^{\mathrm{i}p(y-x)}f(x)\,\mathrm{d}x\,\mathrm{d}p, \\
G^2(t,y)f &= (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\phi(y)t}\,\mathrm{e}^{\mathrm{i}\beta t}\exp[-\mathrm{i}\alpha^1 a_1(y)t] \\
&\quad\times \prod_{k=2}^{3}\exp[\mathrm{i}\alpha^k a_k(y)t]\prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t)\,\mathrm{e}^{\mathrm{i}p(y-x)}f(x)\,\mathrm{d}x\,\mathrm{d}p, \\
&\ \ \vdots \\
G^6(t,y)f &= (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\phi(y)t}\,\mathrm{e}^{\mathrm{i}\beta t}\prod_{k=1}^{3}\exp[-\mathrm{i}\alpha^k a_k(y)t] \\
&\quad\times \prod_{k=1}^{2}\exp(\mathrm{i}\alpha^k p_k t)\exp(-\mathrm{i}\alpha^3 p_3 t)\,\mathrm{e}^{\mathrm{i}p(y-x)}f(x)\,\mathrm{d}x\,\mathrm{d}p.
\end{aligned} \tag{11}$$

If we change the sign in the first exponential matrix in $G(t, y)$ we obtain $G^1(t, y)$, if we change the sign in the first and second we have G^2, etc.

3. Principal results

It is well known that the evolution operator for the Dirac equation (1) conserves the $\mathscr{L}^2$ norm. The same is true for our approximations.

Lemma 1. The operator $E(\pi, \tau)$ is an isometry in $\mathscr{L}^2$, that is,

$$\|E(\pi,\tau)f\| = \|f\| \qquad \text{for all } f\in\mathscr{L}^2. \tag{12}$$

Proof. On account of definition (3), it is enough to prove

$$\|G(t,\cdot)f\| = \|f\| \qquad \text{for all } f\in\mathscr{L}^2. \tag{13}$$

Now we have

$$G(t,y)f = \mathrm{e}^{-\mathrm{i}\phi(y)t}\,\mathrm{e}^{-\mathrm{i}\beta bt}\prod_{k=1}^{3}\exp[\mathrm{i}\alpha^k a_k(y)t]\mathscr{F}^{-1}\left(\prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t)\mathscr{F}(f(x))\right) \tag{14}$$

and (13) follows from proposition 1.

In $C_0^m(\Omega)$ we have the next result.

Lemma 2. Suppose that ϕ, a_k are bounded functions in $C^m(\Omega)$ and have all derivatives up to order m bounded. Then there exists a constant $K = K(m, \tau)$, independent of the partition π, such that K is continuous in τ and

$$\|E(\pi,\tau)f\|_m \leq K\|f\|_m \qquad \text{for all } f\in C_0^m(\Omega). \tag{15}$$

Proof. From (10), (11), (14) and the well known properties of $\mathscr{F}$ we get

$$\frac{\partial G}{\partial y_k}(t,y)f = G(t,y)\frac{\partial f}{\partial x_k} - \mathrm{i}t\frac{\partial \phi}{\partial y_k}G(t,y)f + \mathrm{i}t\sum_{j=1}^{3}\alpha^j\frac{\partial a_j}{\partial y_k}G^j(t,y)f, \tag{16}$$

and in similar manner we obtain

$$D^nG(t,y)f = G(t,y)D^nf + tW \qquad \text{for all } 1\leq |n|\leq m, \tag{17}$$

where W is a sum of functions $G^j(t,y)D^qf$, $0\leq|q|\leq|n|-1$, multiplied by bounded functions. From (13) and (17) we find there exists a positive number C that depends on ϕ, a_k, s and k, such that

$$\|G(t,\cdot)f\|_k \leq \|f\|_k + Ct\|f\|_{k-1} \qquad \text{for } 1\leq k\leq m. \tag{18}$$

Finally we have for $\tau\in(t_{k-1},t_k]$

$$\begin{aligned}\|E(\pi,\tau)f\|_m &\leq \|E(\pi,t_{k-1})f\|_m + C(\tau-t_{k-1})\|E(\pi,t_{k-1})f\|_{m-1}\\ &\leq \|E(\pi,t_{k-2})f\|_m + C(\tau-t_{k-2})\|E(\pi,t_{k-2})f\|_{m-1}\\ &\quad + C^2(\tau-t_{k-1})(t_{k-1}-t_{k-2})\|E(\pi,t_{k-2})f\|_{m-2}\\ &\leq \ldots \leq \|f\|_m + Cs\|f\|_{m-1} + (Cs)^2\|f\|_{m-2} + \ldots + (Cs)^m\|f\|. \end{aligned} \tag{19}$$

Our approximation, as well as the evolution operator for the Dirac equation (1), satisfy:

Lemma 3. If $f\in\mathscr{L}^2(\mathbb{R}^3)$ has support contained in Ω, then $E(\pi,\tau)f$ has support contained in $B(\Omega,\tau)$, where τ is in the time interval $[0,s]$.

Proof. By means of (4), (8) and (9) we obtain

$$\begin{aligned} G(t,y)f &= A\int_{\mathbb{R}^3}\prod_{k=1}^{3}[(2\pi)^{-1/2}\exp(-\mathrm{i}\alpha^k p_k t)\exp(\mathrm{i}p_k y_k)]\hat{f}(p)\,\mathrm{d}p\\ &= \tfrac{1}{8}A\sum_{0\leq n_1,n_2,n_3\leq 1}\int_{\mathbb{R}^3}\bigg(\prod_{k=1}^{3}(2\pi)^{-1/2}(\alpha^k)^{n_k}\{\exp[\mathrm{i}p_k(y_k-t)]\\ &\quad + (-1)^{n_k}\exp[\mathrm{i}p_k(y_k+t)]\bigg)\hat{f}(p)\,\mathrm{d}p\\ &= \tfrac{1}{8}A\sum_{0\leq n_1,n_2,n_3\leq 1}(\alpha^1)^{n_1}(\alpha^2)^{n_2}(\alpha^3)^{n_3}\\ &\quad\times\bigg(\sum_{0\leq k_1,k_2,k_3\leq 1}(-1)^{n_1k_1+n_2k_2+n_3k_3}f(y_1+(-1)^{k_1+1}t, y+(-1)^{k_2+1}t, y\\ &\quad + (-1)^{k_3+1}t)\bigg) \end{aligned} \tag{20}$$

where $A = \mathrm{e}^{-\mathrm{i}\phi(y)t}\,\mathrm{e}^{-\mathrm{i}\beta bt}\,\Pi_{k=1}^{3}\exp[\mathrm{i}\alpha^k a_k(y)t]$. Thus $G(t,y)f$ has support contained in $B(\Omega,t)$ and since $s=\Sigma_{j=1}^{l}(t_j-t_{j-1})$ from (3), (20) the proof of the lemma is completed.

Remark. We can develop (20) in another way

$$G(t, y)f = A\int_{\mathbb{R}^3}\left(\prod_{k=1}^{3}(2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-\mathrm{i}\alpha^k p_k t)\exp[\mathrm{i}p_k(y_k - x_k)]\,\mathrm{d}p_k\right)f(x)\,\mathrm{d}x$$

$$= \tfrac{1}{8}A\int_{\mathbb{R}^3}\left(\sum_{0\leq n_1,n_2,n_3\leq 1}\prod_{k=1}^{3}(\alpha^k)^{n_k}[\delta(y_k - t - x_k)\right.$$

$$\left. + (-1)^{n_k}\delta(y_k + t - x_k)]\right)f(x)\,\mathrm{d}x$$

where $\delta(x)$ is the Dirac delta function. Therefore, if we develop first the integral by p, we obtain the integral with respect to x but only for the paths with the velocity of light c (in this paper we consider the universal constants equal to one). This idea was considered by R P Feynman himself in the one-dimensional case (see Feynman and Hibbs 1965, Rosen 1975). The Feynman approximations are different from our approximation but ours are defined for n dimensions and conserve the $\mathscr{L}^2$ norm.

Lemma 4. Suppose ϕ, a_k are bounded functions. Therefore we have

$$G^j(t,\cdot)\to I \qquad \text{in operator norm as } t\to 0 \tag{21}$$

if $j = 0, 1, \ldots, 6$ where $G^0 = G$ and I is the identity operator in $\mathscr{L}^2$.

Proof. We will prove the lemma only for G. Let $f\in C_0^m(\mathbb{R}^3)$, hence

$$\|(G(t,\cdot) - I)f\|$$

$$= \left\|A\mathscr{F}^{-1}\left(\prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t)\mathscr{F}(f(x)) - \mathscr{F}^{-1}(\mathscr{F}(f(x))\right)\right\|$$

$$= \left\|\prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t)\mathscr{F}(f(x)) - \mathscr{F}(A^{-1}\mathscr{F}^{-1}\mathscr{F}(f(x)))\right\|$$

$$\leq \left\|\prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t) - \mathscr{F}A^{-1}\mathscr{F}^{-1}\right\|\|f(x)\|. \tag{22}$$

Then

$$\|G(t,\cdot) - I\| \leq \left\|\prod_{k=1}^{3}\exp(-\mathrm{i}\alpha^k p_k t) - \mathscr{F}A^{-1}\mathscr{F}^{-1}\right\|; \tag{23}$$

hence we conclude under the assumption of boundedness of ϕ, a_k

$$\|G(t,\cdot) - I\|\to 0 \qquad \text{when } t\to 0.$$

Theorem 1. Suppose ϕ, a_k and their first derivatives are bounded. Moreover, suppose there exists a sequence of partitions π_n, with $\varepsilon(\pi_n)\to 0$, and an operator $E(\tau)$ such that for each $f\in C_0^1(\mathbb{R}^3)$

$$\|E(\pi_n, \tau)f - E(\tau)f\|\to 0 \qquad \text{when } \varepsilon(\pi_n)\to 0 \tag{24}$$

uniformly on $\tau\in[0, s]$. Then $E(\tau)$ is the evolution operator for (1).

Proof. From (4), (10), (11), (14), (16) and the well known properties of $\mathcal{F}$, we get

$$\begin{aligned}\frac{\partial G}{\partial t}(t,y)f &= -\mathrm{i}\beta bG(t,y)f-\mathrm{i}\phi(y)G(t,y)f+\mathrm{i}\sum_{k=1}^{3}\alpha^k a_k(y)G^k(t,y)f\\ &\quad -\sum_{k=1}^{3}\alpha^k G^{k+3}(t,y)\frac{\partial f}{\partial x_k}\\ &= -\mathrm{i}\beta bG(t,y)f-\mathrm{i}\phi(y)G(t,y)f+\mathrm{i}\sum_{k=1}^{3}\alpha^k a_k(y)G(t,y)f\\ &\quad -\sum_{k=1}^{3}\alpha^k G(t,y)\frac{\partial f}{\partial x_k}+\mathrm{i}\sum_{k=1}^{3}\alpha^k a_k(y)\\ &\quad \times[G^k(t,y)f-G(t,y)f]-\sum_{k=1}^{3}\alpha^k[G^{k+3}(t,y)-G(t,y)]\frac{\partial f}{\partial x_k}\\ &= \left(-\mathrm{i}\beta b-\mathrm{i}\phi(y)+\mathrm{i}\sum_{k=1}^{3}\alpha^k a_k(y)-\sum_{k=1}^{3}\alpha^k\frac{\partial}{\partial y_k}\right)G(t,y)f\\ &\quad -\mathrm{i}t\sum_{k=1}^{3}\alpha^k\frac{\partial\phi}{\partial y_k}G(t,y)f+\mathrm{i}t\sum_{k=1}^{3}\alpha^k\left(\sum_{j=1}^{3}\alpha^j\frac{\partial a_j}{\partial y_k}G^j(t,y)f\right)\\ &\quad +\mathrm{i}\sum_{k=1}^{3}\alpha^k a_k(y)[G^k(t,y)f-G(t,y)f]-\sum_{k=1}^{3}\alpha^k[G^{k+3}(t,y)-G(t,y)]\frac{\partial f}{\partial x_k}.\end{aligned} \tag{25}$$

Given $\pi_n=\{t_0,\ldots,t_{l_n}\}$, there exists t_{k_n} such that $\tau\in(t_{k_n},t_{k_n+1}]$. From (3) and (25) we have obviously

$$\begin{aligned}\frac{\partial}{\partial\tau}E(\pi_n,\tau)f &= \left(-\mathrm{i}\beta b-\mathrm{i}\phi(y)+\mathrm{i}\sum_{k=1}^{3}\alpha^k a_k(y)-\sum_{k=1}^{3}\alpha^k\frac{\partial}{\partial y_k}\right)E(\pi_n,\tau)f\\ &\quad -\mathrm{i}(\tau-t_{k_n})\sum_{k=1}^{3}\alpha^k\left(\frac{\partial\phi}{\partial y_k}G(\tau-t_{k_n},y)-\sum_{j=1}^{3}\alpha^j\frac{\partial a_j}{\partial y_k}G^j(\tau-t_{k_n},y)\right)E(\pi_n,t_{k_n})f\\ &\quad +\mathrm{i}\sum_{k=1}^{3}\alpha^k a_k(y)[G^k(\tau-t_{k_n},y)-G(\tau-t_{k_n},y)]E(\pi_n,t_{k_n})f\\ &\quad -\sum_{k=1}^{3}\alpha^k[G^{k+3}(\tau-t_{k_n},y)-G(\tau-t_{k_n},y)]\frac{\partial}{\partial x_k}E(\pi_n,t_{k_n})f.\end{aligned} \tag{26}$$

Now, from (15), (21), (24) and (26) it follows that

$$\lim_{n\to\infty}\frac{\partial}{\partial\tau}E(\pi_n,\tau)f=-\mathrm{i}HE(\tau)f \tag{27}$$

uniformly for $\tau\in[0,s]$, where H is the right-hand side of equation (1). Interchanging the limit (27) and the differentiation we find that $E(\tau)$ is the evolution operator for (1), and this completes the proof.

We will prove our main result.

Theorem 2. Suppose that ϕ, a_k are bounded functions in $C^2(\mathbb{R}^3)$ and have their first derivatives bounded. Then for each f in $\mathscr{L}^2(\mathbb{R}^3)$, the approximations $E(\pi, \tau)f$ converge in $\mathscr{L}^2$ norm to $E(\tau)f$, uniformly for $\tau \in [0, s]$, when $\varepsilon(\pi) = \max(t_{j+1} - t_j)$ converges to zero and where $E(\tau)$ is the evolution operator for the Dirac equation (1).

Proof. In virtue of lemma 1 it is enough to prove the convergence on a dense set of $\mathscr{L}^2(\mathbb{R}^3)$. Let $f \in C_0^\infty(\mathbb{R}^3)$ with support contained in Ω, $\bar{\Omega}$ compact. Under the assumptions, we have from lemma 2

$$\|E(\pi, \tau)f\|_1 \leq K\|f\|_1 \qquad \text{for all } \varepsilon > 0.$$

The last inequality together with proposition 2 and lemma 3 imply that for each sequence of partitions $\{\pi_n\}$, such that $\varepsilon(\pi_n) \to 0$, the set

$$B = \{E(\pi_n, \tau)f : n \to \infty\}$$

is relatively compact in $\mathscr{L}^2(B(\Omega, s))$ for $\tau \in [0, s]$.

From (26) we have that there exists a constant $\mathscr{L} > 0$, independent of n, such that

$$\|(\partial/\partial\tau)E(\pi_n, \tau)f\| \leq \mathscr{L}\|f\|_1 \tag{28}$$

for all $\tau \in [0, s]$, and from Ascoli's theorem, there exists a subsequence $E(\pi_{n(k)}, \tau)f$ that converges uniformly on $[0, s]$.

Finally, we know from theorem 1 and the uniqueness of the solution of the system (1) that all the convergent sequences in $\mathscr{L}^2$ norm of B have the same limit $E(\tau)f$. This implies that $E(\pi_n, \tau)f$ is convergent in norm to $E(\tau)f$, uniformly on $[0, s]$.

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